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Patterns of Symmetry Breaking in QCD at High Baryon Density

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Abstract

We study the structure of QCD at very large baryon density for an arbitrary number of flavors N_f. We provide evidence that for any number of flavors larger than $N_f = 2$ chiral symmetry remains broken at asymptotically large chemical potential. For $N_c = N_f = 3$, chiral symmetry breaking follows the standard pattern $SU(3)_L \times SU(3)_R \to SU(3)$, but for $N_f > 3$ unusual patterns emerge. We study the case $N_f = 3$ in more detail and calculate the magnitude of the chiral order parameters $\langle\bar{\psi}\psi\rangle$ and $\langle(\bar{\psi}\psi)^2\rangle$ in perturbative QCD. We show that, asymptotically, $\langle\bar{\psi}\psi\rangle^{1/3}$ is much smaller than $\langle(\bar{\psi}\psi)^2\rangle^{1/6}$. The result can be understood in terms of an approximate discrete symmetry.

I. INTRODUCTION

The phase structure of matter at non-zero baryon density has recently attracted a great deal of interest. In particular, it has been emphasized that quark matter at very high density is expected to behave as a color superconductor [1, 2]. The behavior of hadronic matter in this regime is of interest in understanding the structure of compact astrophysical objects and the physics of heavy ion collisions in the regime of maximum baryon density. Moreover, it was realized that matter at very high density exhibits many non-perturbative phenomena, such as a mass gap and chiral symmetry breaking, in a regime where the coupling is weak [3, 4]. This means that some of the "hard" problems of QCD can be studied in a systematic fashion.

At very high density the natural degrees of freedom are quasiparticles and holes in the vicinity of the Fermi surface. Since the Fermi momentum is large, asymptotic freedom implies that the interaction between quasiparticles is weak. In QCD, because of the presence of unscreened long range gauge forces, this is not quite true. Nevertheless, we believe that this fact does not essentially modify the argument [5]. However, as we know from the theory of superconductivity the Fermi surface is unstable in the presence of even an arbitrarily weak attractive interaction. At very large density, the attraction is provided by one-gluon exchange between quarks in a color anti-symmetric $\bar{3}$ state. QCD at high density is therefore expected to behave as a color superconductor [6, 7, 8].

Color superconductivity is characterized by the breakdown of color gauge invariance. As usual, this statement has to be interpreted with care. Local gauge invariance cannot really be broken [9]. Nevertheless, spontaneously broken gauge invariance is a useful concept. We can fix the gauge, introduce a gauge non-invariant order parameter and study its effect on gauge invariant correlation functions. The most important gauge invariant consequence of superconductivity is the appearance of a mass gap, through the Meissner-Anderson-Higgs phenomenon. The formation of a mass gap is of course also characteristic of a confined phase. Indeed, it is known that in general Higgs and confined phases are continuously connected [10].

Color superconductivity may also lead to the spontaneous breaking of global symmetries. It is this phenomenon that we wish to study in more detail. Broken global symmetries lead to the appearance of Goldstone bosons, and determine the low energy effective description

of the system. It is one of the remarkable properties of the color superconducting phase in $N_f = 3$ QCD that the pattern of broken global symmetries exactly matches that of QCD at low density. In this work, we would like to establish the global symmetry breaking pattern in QCD with $N_f = 2, 3, \ldots, 6$. In particular, we wish to show that the result in the case $N_f = 3$ is generic in the sense that for any number of flavors larger than 2, chiral symmetry is broken but a vector-like flavor symmetry remains. We also wish to study the most interesting case, QCD with $N_f = 3$, in more detail. We calculate the magnitude of the totally symmetric and anti-symmetric order parameters, and the magnitude of the chiral condensates $\langle\bar{\psi}\psi\rangle$ and $\langle(\bar{\psi}\psi)^2\rangle$. We comment on a number of unusual features of chiral symmetry breaking in QCD at large density.

II. THE EFFECTIVE POTENTIAL

In this section we wish to introduce a simple energy functional that captures the essential dynamics of QCD at high baryon density. We will use this functional in the following section in order to analyze the ground state of QCD with an arbitrary number of flavors.

Our construction is guided by the renormalization group analysis of [11, 12]. In this work we classified possible instabilities of the Fermi surface, and assessed their relative importance, for small and short-range, but otherwise arbitrary couplings near the Fermi surface. It was found that the dominant instability corresponds to scalar diquark condensation. The analysis does not fix the color and flavor channel of this instability uniquely, because there are two equally enhanced interactions. One gluon exchange, which dominates for weak coupling, is attractive in the color anti-symmetric $\bar{3}$ channel, and favors one of these interactions. The same is also true for instanton induced interactions, that are likely to play an important role at moderate densities. As the couplings evolve towards the Fermi surface, the attractive interaction in the color $\bar{3}$ channel will grow, while the repulsive interaction in the color symmetric 6 channel is suppressed. The dominant coupling is a color and flavor anti-symmetric interaction of the form

$$\mathcal{L} \;=\; G(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\left\{\left(\psi_i^a C\gamma_5\psi_j^b\right)\left(\bar{\psi}_k^c C\gamma_5\bar{\psi}_l^d\right) - (C\gamma_5 \leftrightarrow C)\right\}, \qquad (1)$$

where $a, b, \ldots$ are color indices and $i, j \ldots$ are flavor indices. In the following, we shall use the notation $\mathcal{K}_{ijkl}^{abcd}$ for the color-flavor structure of the interaction.

In full QCD, there are unscreened magnetic gluon exchanges and the interaction between quarks is not short range. This problem was first studied in [5]. We expect that this effect does not modify our results for the structure of the ground state, but only the magnitude of the gap. We check this explicitly in the case of $N_f = 3$ in section IV. In this section, we also study the effect of color-symmetric interactions. It was also pointed out that for asymptotically large chemical potential, the gap equation in higher partial wave channels becomes degenerate with the s-wave gap equation [5]. Subleading effects are expected to lift this degeneracy. Using the methods developed in [18] we have checked that the s-wave gap is bigger than higher partial wave gaps. We therefore expect the ground state to be an s-wave superconductor. The only exception is QCD with only one flavor, since in this case a color $\bar{3}$ order parameter cannot have spin zero. This situation is of physical interest for the behavior of real QCD with two light and one intermediate mass flavor. For m_s larger than some critical value, QCD has a phase with separate pairing in the ud and s sectors [13, 14]. In this work, we will not consider the phase structure of $N_f = 1$ QCD.

In order to determine the structure of the ground state we have to calculate the grand canonical potential of the system for different trial states. Since the interaction is attractive in s-wave states, it seems clear that the dominant order parameter is an s-wave, too. We then only have to study the color-flavor structure of the primary condensate. We assume that the condensate takes the form

$$\langle \psi_i^a C\gamma_5 \psi_j^b \rangle = \phi_{ij}^{ab}. \tag{2}$$

ϕ_{ij}^{ab} is a $N_f \times N_f$ matrix in flavor space and a $N_c \times N_c$ matrix in color space. The Pauli principle requires that ϕ_{ij}^{ab} is symmetric under the combined exchange $(ai) \leftrightarrow (bj)$.

We calculate the effective potential using the bosonization method. For this purpose, we introduce collective fields Δ_{ij}^{ab} and $\bar{\Delta}_{ij}^{ab}$ with the symmetries of the order parameter (2). We add to the fermionic action a term $(4G)^{-1}\mathcal{K}_{ijkl}^{abcd}\Delta_{ij}^{ab}\bar{\Delta}_{kl}^{cd}$ and integrate over the dummy variables Δ_{ij}^{ab} and $\bar{\Delta}_{ij}^{ab}$. We then shift the integration variables to eliminate the interaction term (1). So far, no approximations have been made. We now assume that the collective fields are slowly varying, and that Δ_{ij}^{ab} can be replaced by its vacuum expectation value. In this case, we can perform the integration over the fermionic fields and determine the grand canonical potential as a function of the gap matrix Δ_{ij}^{ab}.

The integration over the fermions is performed using the Nambu-Gorkov formalism. We

introduce a two component field $\Psi = (\psi, \bar{\psi}^T)$. The inverse quark propagator takes the form

$$S^{-1}(q) = \begin{pmatrix} \not{q} + \not{\mu} - m & \mathcal{K} \cdot \bar{\Delta} \\ \mathcal{K} \cdot \Delta & (\not{q} - \not{\mu} + m)^T \end{pmatrix}. \tag{3}$$

The grand canonical potential is now given by

$$\Omega(\Delta) = \frac{1}{2} \mathrm{Tr} \left[\log(S_0^{-1} S) \right] + \frac{1}{4G} \Delta \cdot \mathcal{K} \cdot \bar{\Delta}. \tag{4}$$

In order to evaluate the logarithm, we have to diagonalize the mass matrix $\mathcal{M} = \mathcal{K} \cdot \Delta$. Let us denote the corresponding eigenvalues by $\Delta_\rho \, (\rho = 1, \ldots, N_c N_f)$. These are the physical gaps of the $N_f N_c$ fermion species. If we neglect the quark masses m, the grand potential is

$$\Omega(\Delta) = \sum_\rho \left\{ - \int \frac{d^3p}{(2\pi)^3} \left(\sqrt{(p-\mu)^2 + \Delta_\rho^2} + \sqrt{(p+\mu)^2 + \Delta_\rho^2} \right) + \frac{1}{G} \Delta_\rho^2 \right\} \tag{5}$$

The momentum integral has an ultraviolet divergence. This integral can be regularized by introducing a cutoff Λ or, more generally, by including a form factor $F(p)$. In this case it would seem that the properties of the grand potential depend on a number of parameters, such as the coupling constant G, the chemical potential μ and the cutoff Λ. In the weak coupling limit, however, the grand potential should only depend on the value of the gap on the Fermi surface, and not on the exact momentum dependence of the interaction. This can be made manifest by introducing a renormalized grand potential. Following the work of Weinberg [15], we have

$$\Omega_{ren}(\Delta) = \sum_\rho \left\{ \frac{\mu^2}{4\pi^2} \Delta_\rho^2 \left(\log\left(\frac{\Delta_\rho}{\xi} \right) - 1 \right) + \frac{1}{G(\xi)} \Delta_\rho^2 \right\}. \tag{6}$$

Here, ξ is a renormalization scale. The grand potential is independent of ξ, since the scale dependence of the first term is canceled by the scale dependence of the coupling constant G. The coupling constant satisfies the renormalization group equation discussed in [11, 12].

The grand potential (6) depends on $N_c(N_c-1)N_f(N_f-1)/4$ parameters. We minimize this function numerically. After we determine the matrix Δ_{ij}^{ab} that minimizes the grand potential we study the corresponding symmetry breaking pattern. Without superconductivity, there are $N_f^2 - 1$ global flavor symmetries for both left and right handed fermions, as well as $N_c^2 - 1$ local gauge symmetries. Superconductivity reduces the amount of symmetry. In order to find the residual symmetry group we study the second variation of the order parameter $\delta^2\Delta/(\delta\theta_i\delta\theta_j)$, where θ_i $(i = 1, \ldots, N_f^2 + N_c^2 - 2)$ parameterizes the flavor and color transformations. Zero eigenvalues of this matrix correspond to unbroken color-flavor symmetries.

III. COLOR SUPERCONDUCTIVITY IN QCD WITH $N_c = 3$ COLORS AND N_f FLAVORS

The results of our numerical study for QCD with three colors and $N_f = 2, \ldots, 6$ flavors are summarized in Table 1. In the following, we will study each of these cases in more detail.

The two flavor case is special. In this case, the order parameter

$$\Delta_{ij}^{ab} = \Delta \epsilon^{3ab} \epsilon_{ij} \tag{7}$$

only breaks color $SU(3) \to SU(2)$. The chiral $SU(2)_L \times SU(2)_R$ symmetry remains unbroken. At this level, the Fermi surfaces of the up and down quarks of the third color remain intact. A careful study of the quantum numbers of the low energy states shows that chiral symmetry is realized in terms of massless protons and neutrons [13]. The proton and neutron are composites of the quark and Higgs fields. Subleading interactions can generate a gap for these states. The exact nature of this gap is hard to determine, even in the limit of very large chemical potential.

For N_f larger than 2, the gauge symmetry is always completely broken, and all quarks acquire a gap. In addition to that, we find that the preferred order parameter always involves a coupling between the color and flavor degrees of freedom. This means that the original flavor symmetry is broken, but some vector-like symmetry which is a combination of the original flavor and color symmetries remains.

In the case of three flavors we find that the preferred order parameter is of the form

$$\Delta_{ij}^{ab} = \Delta(\delta_i^a \delta_j^b - \delta_j^a \delta_i^b). \tag{8}$$

This is the color-flavor locked phase suggested in [3]. Both color and flavor symmetry are completely broken. There are eight combinations of color and flavor symmetries that generate unbroken global symmetries. The symmetry breaking pattern is

$$SU(3)_L \times SU(3)_R \times U(1)_V \to SU(3)_V. \tag{9}$$

This is exactly the same symmetry breaking that QCD exhibits at low density. This has led to the idea that in QCD with three flavors, the low and high density phases might be continuously connected [4]. We also note that the quark mass gaps fall into representations ([8]+[1]) of the unbroken vector symmetry. Note that in the present analysis, which only

takes into account the leading interaction, the order parameter is completely anti-symmetric in both color and flavor. In the case $N_f = N_c$, however, there is a more general order parameter $\Delta^{ab}_{ij} = \Delta_1 \delta^a_i \delta^b_j + \Delta_2 \delta^a_j \delta^b_i$ which has both symmetric and anti-symmetric components and has the same residual symmetry. We will discuss this situation in more detail in the next section.

In the case $N_f = 4$ the numerical results indicate that the most favorable order parameter is given by

$$\Delta^{ab}_{ij} = \Delta \epsilon^{abc} \eta^c_{ij} = \Delta \epsilon^{abc} (\epsilon_{ijc} + \delta_{ic}\delta_{j4} - \delta_{jc}\delta_{i4}), \tag{10}$$

where η^a_{ij} is the 't Hooft symbol. There is a second, degenerate, solution where we replace $\eta^a_{ij} \to \bar{\eta}^a_{ij} = \epsilon_{ija} - \delta_{ia}\delta_{j4} + \delta_{ja}\delta_{i4}$. The 't Hooft symbol describes the isomorphism $SU(2) \times SU(2) \equiv O(4)$. Consider the $O(4)$ generators $(M^{\rho\sigma})_{\mu\nu} = i(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\sigma_\mu \delta^\rho_\nu)$. The $O(4)$ generators can be decomposed into $SU(2)_L \times SU(2)_R$ generators $\eta^a_{\mu\nu}(M^{\mu\nu})_{\rho\sigma}$ and $\bar{\eta}^a_{\mu\nu}(M^{\mu\nu})_{\rho\sigma}$. Using this result one can show that the $N_f = 4$ order parameter (10) realizes the symmetry breaking pattern

$$SU(4)_L \times SU(4)_R \times U(1)_V \to SU(2)_V \times SU(2)_V. \tag{11}$$

The generators of the $SU(2) \times SU(2)$ symmetry are

$$\left(\epsilon^{abc} N_{bc} + \frac{1}{2} \eta^a_{\mu\nu} M^{\mu\nu} \right), \quad \left(\epsilon^{abc} N_{bc} + \frac{1}{2} \bar{\eta}^a_{\mu\nu} M^{\mu\nu} \right). \tag{12}$$

Here, N_{bc} are $O(3) \subset SU(3)$ color generators and $M_{\mu\nu}$ are $O(4) \subset SU(4)$ flavor generators. We note that for $N_f = 4$, and indeed for any N_f larger than three, we cannot realize the same symmetry breaking that we have at low density. This means that even though chiral symmetry remains broken at very large density, there has to be a phase transition that separates the high and low density phases.

In the case $N_f = 5$ we were unable to find a compact expression for the energetically favored order parameter. The numerical results indicate that $N_f = 5$ QCD realizes the symmetry breaking pattern

$$SU(5)_L \times SU(5)_R \times U(1)_V \to SU(2)_V. \tag{13}$$

The residual symmetry is smaller than for any other number of flavors. This is also reflected in the fact that the condensation energy is comparably small.

If N_f is a multiple of N_c, the dominant gap corresponds to multiple embeddings of the $N_f = N_c$ order parameter. For $N_f = 6$, we have

$$\Delta_{ij}^{ab} = \Delta\epsilon^{abc}(\epsilon_{ijc} + \epsilon_{(i-3)(j-3)c}). \tag{14}$$

This order parameter corresponds to the symmetry breaking pattern

$$SU(6)_L \times SU(6)_R \times U(1)_V \to SU(3)_V \times U(1)_V \times U(1)_A. \tag{15}$$

The $SU(3)$ symmetry is a double embedding of the $SU(3)$ that appear in the $N_f = 3$ color-flavor locked phase. The original $U(1)_V$ and approximate $U(1)_A$ symmetries are broken, but a new $U(1)_V \times U(1)_A$ appears. This symmetry is a subgroup of the original $SU(6)_L \times SU(6)_R$ flavor symmetry which rotates the two 3×3 blocks in equation (14). The extra $U(1)$ symmetries may be broken by higher order condensates.

In this section we have analyzed the color-flavor structure of the superconducting ground state using a BCS-like energy functional. Another method, which has proven to be very useful in the context of liquid ^3He and other systems [16], is the Landau-Ginzburg functional. This method was applied to color superconductivity in [8, 17]. In this case we construct the most general energy functional that is consistent with the symmetries of QCD and a simple polynomial in the order parameter. In four dimensions, it is usually sufficient to keep terms that are at most quartic in the fields. We can analyze the possible ground states by studying the minima of the Landau-Ginzburg functional.

There is one restriction that one has to keep in mind. At $T = 0$ the free energy of the system in not an analytic function of the gap so that, strictly speaking, the free energy cannot be expanded as a power series in the order parameter. Only in the vicinity of the finite temperature phase transition does the expansion in powers of the order parameter have a firm foundation.

Nevertheless, the Landau-Ginzburg description is very useful in describing a wealth of phenomena, even at $T = 0$. If we restrict ourselves to color and flavor anti-symmetric order parameters we can represent the order parameter matrix Δ_{ij}^{ab} by the field ϕ_i^a, where a is an index in the anti-symmetric $[N_c(N_c - 1)/2]$ of color and i in the $[N_f(N_f - 1)/2]$ of flavor. Color and flavor invariance imply that the Landau-Ginzburg effective potential has the form

$$V = -m^2 \mathrm{tr}(\phi^\dagger\phi) + \lambda_1 \Big[\mathrm{tr}(\phi^\dagger\phi)\Big]^2 + \lambda_2 \mathrm{tr}\Big[(\phi^\dagger\phi)^2\Big], \tag{16}$$

where $(\phi^\dagger\phi)^{ab} = (\phi_i^a)^*\phi_i^b$, and m^2, λ_1 and λ_2 are arbitrary parameters. Of course, we can equally well write (16) in terms of $(\phi\phi^\dagger)_{ij} = \phi_i^a(\phi_j^a)^*$. Note that there are no $\det(\phi)$ terms, because such a term would violate the $U(1)$ of baryon number. The effective potential only depends on the eigenvalues of $\phi^\dagger\phi$. This means that the effective potential only depends on a number of parameters, $N_c(N_c-1)/2$ (or $N_f(N_f-1)/2$, whichever is smaller).

For $N_f = 2$ there is only one quartic term and the minimum occurs for $\phi^a = \Delta\delta^{a3}$, as we would expect. In the case $N_f = 3$ there are two possible minima, $\phi_i^a = \Delta\delta_i^a$ and $\phi_i^a = \Delta\delta^{a3}\delta_{i3}$. These minima correspond to the color-flavor locked phase and the $N_f = 2$ phase. The groundstate depends on the values of the coupling constants. If $\lambda_2 > 0$, the ground state is the color-flavor locked phase. Extending the mean field description described in the last section to $T \neq 0$, we can check that indeed $\lambda_1 = 0$ and $\lambda_2 > 0$, see also [8, 17].

For $N_f > 3$ the effective potential does not depend on the number of flavors. This means that there are a large number of ground states all of which are degenerate with an embedding of the $N_f = 3$ color-flavor locked phase. The ground states of the BCS functional discussed above are minima of the Landau-Ginzburg potential, but the effective potential (16) does not distinguish partially gapped from fully gapped states. In order to construct an effective energy functional for which the fully gapped state is the true minimum one has to include higher powers in the order parameter. This, of course, would also introduce additional parameters and we will not pursue this problem here.

IV. MORE ON COLOR-FLAVOR-LOCKING IN QCD WITH $N_c = N_f = 3$

In this section we wish to examine the color-flavor-locked state in QCD with three colors and flavors in somewhat more detail. For this purpose, we will concentrate on the regime of very large chemical potential, $\mu \gg \Lambda_{QCD}$, in which perturbative calculations are possible. The determination of the gap in perturbative QCD has recently attracted some attention [5, 18, 19, 20, 21, 22]. The main conclusion is that the gap is dominated by almost collinear magnetic gluon exchanges. The magnitude of the gap is

$$\Delta_0 \simeq 256\pi^4(2/N_f)^{5/2}\mu g^{-5}\exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right). \tag{17}$$

We should emphasize that, strictly speaking, this result contains only an estimate of the preexponent. This estimate is obtained by collecting the leading logarithms from electric

and magnetic gluon exchanges [18]. There are corrections of order $O(1)$ that originate from improved matching of the gap function at $p_0 \simeq \Delta_0$ and $p_0 \simeq g\mu$, a self-consistent treatment of the Meissner effect, and, possibly, higher order perturbative corrections. Nevertheless, the main point is that (17) is the result of a well-defined calculation that can be systematically improved.

In previous works, the gap equation was always studied for a one-component color-flavor anti-symmetric order parameter. This is appropriate for QCD with two flavors, but in the case of more than two flavors the gap equation is more complicated. In the previous section we studied a more general $N_cN_f \times N_cN_f$ dimensional gap matrix but restricted ourselves to short range, color and flavor antisymmetric interactions. The restriction to antisymmetric gap matrices is probably justified for $N_f \neq N_c$, but for $N_f = N_c$ the symmetric and anti-symmetric order parameters have the same global symmetry, so there is no symmetry reason for the symmetric gap parameter to be zero.

In three flavor QCD, the order parameter has the form

$$\Delta_{ij}^{ab} = \Delta_A(\delta_i^a\delta_j^b - \delta_j^a\delta_i^b) + \Delta_S(\delta_i^a\delta_j^b + \delta_j^a\delta_i^b). \tag{18}$$

We can now repeat the perturbative calculation of the gap using this particular ansatz for the order parameter. We will follow the method described in [18]. Just as in the $N_f = 2$ case, the gap depends on frequency and the Dirac structure of the gap matrix is proportional to $C\gamma_5(1+\vec{\alpha}\cdot\hat{p})/2$. The only new ingredient is that we have to take into account the more complicated color-flavor structure when we calculate the Nambu-Gorkov propagator. As in the last section, this is most easily accomplished by viewing (18) as a matrix in a N_cN_f dimensional color-flavor space. The eigenvalues of this matrix are

$$\Delta_8 = \Delta_A - \Delta_S, \qquad \Delta_1 = 2\Delta_A + 4\Delta_S, \tag{19}$$

where the subscript indicates the degeneracy. For three degenerate flavors the normal components of the inverse Nambu-Gorkov propagator are proportional to the unit matrix in color-flavor space, so they remain diagonal as the anomalous components are diagonalized. We can now determine the propagator by inverting the Nambu-Gorkov and Dirac structure as in the $N_f = 2$ case. The off-diagonal propagator S_{21} needed in the gap equation is a diagonal matrix in color-flavor space with entries

$$S_{21}^i(q) = \frac{1}{2}(C\gamma_5)\frac{\Delta_i(1-\vec{\alpha}\cdot\vec{q})}{q_0^2-(q-\mu)^2-\Delta_i^2}, \tag{20}$$

with $\Delta_i = (\Delta_8, \Delta_8, \ldots, \Delta_1)$ as in (19). Having determined the propagator we have to calculate the color factor. It is given by

$$c_A = \frac{1}{4}(\lambda^A)^T_{ab}(\delta^a_i\delta^b_j - \delta^b_i\delta^a_j)(\lambda^A)_{cd} = -\frac{N_c+1}{2N_c}(\delta^c_i\delta^d_j - \delta^d_i\delta^c_j) \tag{21}$$

for the color-flavor anti-symmetric gap and

$$c_S = \frac{1}{4}(\lambda^A)^T_{ab}\left(\delta^a_i\delta^b_j + \delta^b_i\delta^a_j\right)(\lambda^A)_{cd} = \frac{N_c-1}{2N_c}\left(\delta^c_i\delta^d_j + \delta^d_i\delta^c_j\right) \tag{22}$$

for the symmetric gap. The anti-symmetric color factor c_A agrees with the result for the $N_f = 2$ order parameter. We can now project the gap equation on the color-flavor anti-symmetric and symmetric structures. We find

$$\begin{aligned}\Delta_A(p_0) &= \frac{g^2}{18\pi^2}\int dq_0 \log\left(\frac{b\mu}{|p_0-q_0|}\right)\left\{\frac{2}{3}\frac{\Delta_A(q_0)-\Delta_S(q_0)}{\sqrt{q_0^2+(\Delta_A(q_0)-\Delta_S(q_0))^2}}\right.\\ &\qquad\left. + \frac{1}{6}\frac{2\Delta_A(q_0)+4\Delta_S(q_0)}{\sqrt{q_0^2+(2\Delta_A(q_0)+4\Delta_S(q_0))^2}}\right\}, \end{aligned}\tag{23}$$

$$\begin{aligned}\Delta_S(p_0) &= \frac{g^2}{18\pi^2}\int dq_0 \log\left(\frac{b\mu}{|p_0-q_0|}\right)\left\{\frac{1}{6}\frac{\Delta_A(q_0)-\Delta_S(q_0)}{\sqrt{q_0^2+(\Delta_A(q_0)-\Delta_S(q_0))^2}}\right.\\ &\qquad\left. - \frac{1}{12}\frac{2\Delta_A(q_0)+4\Delta_S(q_0)}{\sqrt{q_0^2+(2\Delta_A(q_0)+4\Delta_S(q_0))^2}}\right\}, \end{aligned}\tag{24}$$

where $b = 256\pi^4(2/N_f)^{5/2}g^{-5}$. This is a complicated system of coupled equations, but the situation simplifies in the weak coupling limit. In this case, we can assume that $\Delta_S \ll \Delta_A$. The equation for Δ_A then becomes

$$\Delta_A(p_0) = \frac{g^2}{18\pi^2}\int dq_0 \log\left(\frac{b\mu}{|p_0-q_0|}\right)\left\{\frac{2}{3}\frac{\Delta_A(q_0)}{\sqrt{q_0^2+\Delta_A(q_0)^2}} + \frac{1}{3}\frac{\Delta_A(q_0)}{\sqrt{q_0^2+(2\Delta_A(q_0))^2}}\right\}. \tag{25}$$

If it were not for the factor 2 in the denominator of the second term in the curly brackets, this would be exactly the same equation as the one we found for the simple $N_f = 2$ order parameter. We can take the factor 2 into account in an approximate way by rescaling the integration variable in the second term. In this way we can reduce equation (25) to the gap equation for the $N_f = 2$ order parameter with the coefficient b rescaled by a factor $2^{-1/3}$. This means that

$$\Delta_A \simeq 2^{-1/3}256\pi^4(2/N_f)^{5/2}\mu g^{-5}\exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right). \tag{26}$$

The equation for Δ_S can be analyzed in a similar fashion. We find

$$\Delta_S \simeq \frac{g}{\pi}\frac{\sqrt{2}\log(2)}{36}\Delta_A. \tag{27}$$

This implies that, formally, Δ_S is suppressed by one power of the coupling constant, g. In addition to that, we note that the numerical coefficient in (27) is quite small, so that $\Delta_S \ll \Delta_A$ even if g is not small.

These results are easily generalized to an arbitrary number of flavors and colors with $N_f = N_c = N$. The eigenvalues of the gap matrix are

$$\Delta_{N^2-1} = \Delta_A - \Delta_S, \qquad \Delta_1 = (N-1)\Delta_A + (N+1)\Delta_S, \tag{28}$$

and the N_c dependence of the anti-symmetric and symmetric color factor is given in (21,22). From these results we find that the color-flavor-locked gap is given by

$$\Delta_A = 256\pi^4(N-1)^{-\frac{N-1}{2N}}(2/N)^{5/2}\mu g^{-5}\exp\left(-\frac{\pi^2}{g}\sqrt{\frac{6N}{N+1}}\right), \tag{29}$$

$$\Delta_S = \frac{g}{6\pi}\left(\frac{N-1}{2N}\right)^2\left(\frac{6N}{N+1}\right)^{1/2}\log(N-1)\Delta_A. \tag{30}$$

The origin of the various factors of N in (29) is easily explained. The factor in the exponent is the color factor that comes from the tree level scattering amplitude of two quarks in a color anti-symmetric state. The factor $N^{-5/2}$ originates from the flavor dependence of the screening mass, and the factor $(N-1)$ raised to the power $-(N-1)/(2N)$ comes from the structure of the color-flavor locked state. This factor implies that for large N, the CFL gap is suppressed by a factor $\sqrt{N}$ with respect to the gap in the $N_f = 2$ phase. As a consequence, for $N > 3$ the color-flavor locked state (18) is not necessarily the energetically favored state. If, on the other hand, we take the large N_c limit with $N_f = 3$ fixed, we find multiple embeddings of the $N_c = N_f = 3$ state. If the large N_c limit is taken with the conventional scaling $g^2 N_c = const$, the superconducting gap is suppressed by $\exp(-\sqrt{N_c})$. This means that for very large N_c and N_f fixed, the superconducting ground state is disfavored compared to a chiral density wave [23, 24].

In addition to the gap equation, we would also like to study the condensation energy. To order g^2 the grand potential can be calculated from [8, 25]

$$\Omega = \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left\{-\mathrm{tr}\left[S(q)\Sigma(q)\right] + \mathrm{tr}\log\left[S_0^{-1}(q)S(q)\right]\right\}, \tag{31}$$

where $S(q)$ and $\Sigma(q)$ are the Nambu-Gorkov propagator and proper self energy given in [18]. The grand potential has the form $\Omega = 8f(\Delta_8) + f(\Delta_1)$ where $\Delta_{8,1}$ are the singlet and octet gaps. The functional $f(\Delta)$ is given by

$$f(\Delta) = \frac{\mu^2}{\pi^2}\int dp_0 \left\{ -\frac{\Delta(p_0)}{\sqrt{p_0^2+\Delta(p_0)^2}} + \sqrt{p_0^2+\Delta(p_0)^2} - p_0 \right\}. \tag{32}$$

This result is very similar to the result in the case of short range interactions [8]. The only difference is that the energy dependence of the gap function acts as an explicit cutoff in the integrals. We can calculate the integrals using the approximate solution of the Eliashberg equation derived by Son [5]. Numerically we find that the condensation energy scales to very good accuracy as

$$f(\Delta) = \frac{\mu^2}{4\pi^2}\Delta_0^2 \log\left(\frac{\Delta_0}{\mu}\right), \tag{33}$$

where $\Delta_0 = \Delta(p_0 = 0)$. This is very similar to the result we found for short range interactions, equation (6). This suggests that the results obtained in section III remain valid in the more general case of long range interactions. In particular, we can calculate the energy gain of the color-flavor locked state over the $N_f = 2$ state. The result is only very weakly dependent on the gap in a very wide range of Δ/μ. In the weak coupling limit, we find $\epsilon(CFL)/\epsilon(N_f\!=\!2) \simeq 1.9$. This ratio is somewhat smaller than what one would expect based on the number of gaps, $9/4 \simeq 2.2$.

V. CHIRAL SYMMETRY BREAKING

The most interesting aspect of the color-flavor locked phase is that chiral symmetry is broken, and that the form of the corresponding low energy effective action agrees with QCD at low density [4, 26]. But while the coefficients of the effective lagrangian are complicated, non-perturbative quantities in QCD at low density, they can be calculated perturbatively at high density.

In this section we would like to begin this program by calculating the chiral order parameter in the color-flavor locked phase. As a first step, we have to calculate the superfluid condensate. For a single fermion species we have

$$\phi = \langle \psi C\gamma_5 \frac{1}{2}(1+\vec{\alpha}\cdot\hat{p})\psi\rangle = \frac{\mu^2}{\pi^2}\int dp_0 \frac{\Delta(p_0)}{\sqrt{p_0^2+\Delta(p_0)^2}}. \tag{34}$$

In the weak coupling limit, the integrand can be written as a total derivative using the differential equation for $\Delta(p_0)$ [5, 27]. The result is

$$\phi = 2\left(\frac{\mu^2}{2\pi^2}\right)\frac{3\sqrt{2}\pi}{g}\Delta. \tag{35}$$

In the CFL state, the color-flavor structure of the condensate is more complicated. Just like the gap matrix, the condensate can be written as

$$\langle\psi_i^a C\gamma_5\frac{1}{2}(1+\vec{\alpha}\cdot\hat{p})\psi_j^b\rangle = \phi_A(\delta_i^a\delta_j^b-\delta_j^a\delta_i^b)+\phi_S(\delta_i^a\delta_j^b+\delta_j^a\delta_i^b). \tag{36}$$

The two condensates $\phi_{A,S}$ can be determined from the octet and singlet gap parameters. Not surprisingly, we find that ϕ_S is small in the weak coupling limit. In the case of ϕ_A, all color and flavor factors drop out and

$$\phi_A = 2\left(\frac{\mu^2}{2\pi^2}\right)\frac{3\sqrt{2}\pi}{g}\Delta. \tag{37}$$

The chiral structure of the superfluid order parameter is $\psi C\gamma_5\psi = \psi_R\psi_R - \psi_L\psi_L$. This means that pairing takes place between quarks of the same chirality. A convolution of two superfluid order parameters $(\psi_R\psi_R)(\bar\psi_L\bar\psi_L)$ will then directly yield the gauge invariant order parameter $\langle(\bar\psi_L\psi_R)(\bar\psi_L\psi_R)\rangle$. In the weak coupling limit, factorization is valid and we find

$$\langle(\bar\psi_L\psi_R)(\bar\psi_L\psi_R)\rangle = \frac{1}{4}\left(12\phi_A^2+24\phi_S^2\right)\simeq 3\phi_A^2. \tag{38}$$

In the same way we can calculate many other four fermion operators, like

$$\langle(\bar\psi\psi)^2\rangle = \langle(\bar\psi_L\psi_R)(\bar\psi_L\psi_R)\rangle + \langle(\bar\psi_R\psi_L)(\bar\psi_R\psi_L)\rangle = 6\phi_A^2. \tag{39}$$

Here, we have used $\langle(\bar\psi_L\psi_R)(\bar\psi_R\psi_L)\rangle \simeq 0$. This is a consequence of the chiral structure of the superfluid order parameter mentioned above.

There is an important point that we need to emphasize here. The vacuum expectation value $\langle(\bar\psi\psi)^2\rangle$ is not an order parameter for chiral symmetry breaking. This is most obvious in the case of two flavors. In this case, $(\bar\psi\psi, \bar\psi i\gamma_5\vec{\tau}\psi)$ transforms as a 4-vector under the chiral $SU(2)_L\times SU(2)_R = O(4)$. Chiral symmetry restoration then implies $3\langle(\bar\psi\psi)^2\rangle = \langle(\bar\psi i\vec{\tau}\gamma_5\psi)^2\rangle$, not $\langle(\bar\psi\psi)^2\rangle = 0$. In the case of $N_f = 3$, a true order parameter for chiral symmetry breaking is given by [28]

$$\mathcal{O}_1 = (\bar\psi_L\gamma_\mu\lambda^a\psi_L)(\bar\psi_R\gamma_\mu\lambda^a\psi_R), \tag{40}$$

where λ^a is a flavor generator. This operator transforms as $(8,8)$ under $SU(3)_L \times SU(3)_R$, so a non-zero expectation value of $\mathcal{O}_1$ definitely implies that chiral symmetry is broken. Fierz rearranging $\mathcal{O}_1$, we get an alternative order parameter

$$\mathcal{O}_2 = \frac{2(N_f^2-1)}{N_f^2}(\bar\psi_L\psi_R)(\bar\psi_R\psi_L) - \frac{1}{N_f}(\bar\psi_L\lambda^a\psi_R)(\bar\psi_R\lambda^a\psi_L). \tag{41}$$

We can calculate the vacuum expectation values of $\mathcal{O}_{1,2}$ in the mean field approximation. Both turn out to be zero. This does not, of course, imply that chiral symmetry is unbroken. The fact that $\langle\mathcal{O}_{1,2}\rangle$ vanishes is due to an accidental symmetry of the mean field approximation. The superfluid order parameter is invariant under $(Z_2)_L \times (Z_2)_R$ symmetries that act on the fermion and anti-fermion fields separately. Because $\mathcal{O}_{1,2}$ do not have this symmetry, they cannot acquire an expectation value in the mean field approximation. This $(Z_2 \times Z_2)^2$ is not a symmetry of QCD. As a result, we expect that $\mathcal{O}_{1,2}$ will develop an expectation value once higher order perturbative corrections are included. Another way to look at this phenomenon is the observation that the color-flavor locked order parameter couples left and right handed fields only indirectly, through the vector-like character of the gauge symmetry. A non-vanishing expectation value for an operator of the form $(\bar LL)(\bar RR)$ only arises at higher order in perturbation theory.

Since chiral symmetry is broken, we also expect that the standard chiral order parameter $\langle\bar\psi\psi\rangle$ acquires an expectation value. It was noted in [3, 29] that the color-flavor locked phase has an approximate $(Z_2)_L \times (Z_2)_R$ symmetry. If this symmetry were exact, then the quark condensate would be zero. In QCD instantons break $(Z_2)_L \times (Z_2)_R$ to $(Z_2)_V$ and lead to a non-vanishing quark condensate. In [29] we calculated the quark condensate at moderate densities, assuming that instantons dominate not only the quark condensate, but also the superfluid gap. At very large density, instantons are suppressed and the gap equation is dominated by perturbative effects. The results of [29] are easily generalized to this case.

In QCD with three flavors, the instanton induced interaction between quarks is given by [30, 31, 32]

$$\begin{aligned}\mathcal{L} \;=\; & G\frac{1}{6N_c(N_c^2-1)}\epsilon_{f_1f_2f_3}\epsilon_{g_1g_2g_3}\left(\frac{2N_c+1}{2N_c+4}(\bar\psi_{L,f_1}\psi_{R,g_1})(\bar\psi_{L,f_2}\psi_{R,g_2})(\bar\psi_{L,f_3}\psi_{R,g_3})\right. \\ & \left. +\frac{3}{8(N_c+2)}(\bar\psi_{L,f_1}\psi_{R,g_1})(\bar\psi_{L,f_2}\sigma_{\mu\nu}\psi_{R,g_2})(\bar\psi_{L,f_3}\sigma_{\mu\nu}\psi_{R,g_3}) + (L\leftrightarrow R)\right).\end{aligned} \tag{42}$$

The coupling constant G is determined by a perturbative calculation of small fluctuations

around the classical instanton solution. The result is

$$G = \int d\rho\; n(\rho,\mu)\; (2\pi\rho)^6, \tag{43}$$

with

$$\begin{aligned} n(\rho,\mu) &= C_N \left(\frac{8\pi^2}{g^2}\right)^{2N_c} \rho^{-5} \exp\left[-\frac{8\pi^2}{g(\rho)^2}\right] \exp\left[-N_f\rho^2\mu^2\right], \\ C_N &= \frac{1}{(N_c-1)!(N_c-2)!}\, 0.466 \exp(-1.679N_c) 1.34^{N_f}. \end{aligned} \tag{44}$$

At zero density, the ρ integral in (43) is divergent at large ρ. This is the famous infrared problem of the semi-classical approximation in QCD. At large chemical potential, however, everything is under control and G is reliably determined. We find

$$G = \frac{1}{2}\frac{(2\pi)^6}{6N_c(N_c^2-1)} C_N S_0^{2N_c} \Lambda^{-5} N_f^{-\frac{5+b}{2}} \left(\frac{\Lambda}{\mu}\right)^{5+b} \Gamma\left(\frac{b+5}{2}\right). \tag{45}$$

Here, Λ is the QCD scale parameter, $b = \frac{11}{3}N_c - \frac{2}{3}N_f = 9$ is the first coefficient of the QCD beta function and $S_0 = 8\pi^2/g^2$. The result shows that, asymptotically, the coupling constant G has a very strong power-law suppression $\sim \mu^{-(5+b)} = \mu^{-14}$.

Since G is so small, we can treat the effect of instantons as a perturbation. In the color-flavor locked phase the instanton vertex induces a fermion mass term. This can be seen by saturating four of the external legs of the interaction (42) with the superfluid order parameter (36). Using the results of [29], we find

$$\mathcal{L} = (\bar\psi^a_{L,i}\frac{1}{2}(1+\vec\alpha\cdot\hat p)\psi^b_{R,j}) \left\{\left(\delta^{ab}\delta_{ij}\right)\frac{18}{5} + \left(\delta^a_i\delta^b_j\right)\frac{6}{5}\right\} G\phi_A^2 + (L\leftrightarrow R). \tag{46}$$

We note that there are two kinds of fermion mass terms. Both the color singlet structure $(\delta^{ab}\delta_{ij})$ and the color octet structure $(\delta^a_i\delta^b_j)$ are compatible with the residual $SU(3)_V$ symmetry of the color-flavor locked phase. From the result (46), we can directly read off the singlet and octet mass terms

$$m_0 = \frac{18}{5}G\phi_A^2, \qquad m_8 = \frac{6}{5}G\phi_A^2. \tag{47}$$

From these results we can also determine the quark condensate. For this purpose we have to know the momentum dependence of the fermion propagator in the vicinity of the Fermi surface. In principle, this is determined by the momentum dependence of the $\mu \neq 0$ fermion

zero mode of the instanton, see [29]. For simplicity, we assume that the quark mass has the same momentum dependence as the gap. In this case, we find

$$\langle\bar\psi\psi\rangle = -2\left(\frac{\mu^2}{2\pi^2}\right)\frac{3\sqrt{2}\pi}{g}\frac{18}{5}G\phi_A^2. \tag{48}$$

From the instanton calculation we can also see how to construct a chiral order parameter that is non-vanishing already on the level of the mean field approximation. Consider the 8-fermion operator

$$\mathcal{O}_8 = \det(\bar\psi_L\psi_R)\bar\psi_L\psi_R, \tag{49}$$

where $\det(\bar\psi_L\psi_R)$ is a short hand expression for the flavor structure of the instanton vertex (42). The determinant is invariant under $SU(3)\times SU(3)_R$, so $\mathcal{O}_8$ transforms like the quark condensate. From the discussion above it is clear that $\mathcal{O}_8$ acquires an expectation value in the color-flavor locked phase. We find

$$\langle\mathcal{O}_8\rangle = 6\phi_A^4. \tag{50}$$

There is nothing special about $\mathcal{O}_8$, other eight-fermion operators are equally good order parameters.

Finally, it is interesting to obtain a few numerical estimates. For definiteness, we will consider the chemical potential $\mu = 500$ MeV. Also, we will use $\Lambda_{QCD} = 200$ MeV. Following [19] we will be optimistic and use $g = 4.2$, which corresponds to the maximum of the gap as a function of g. In this case we get a substantial gap in QCD with two flavors, $\Delta_0(N_f = 2) = 130$ MeV. The gap in the $N_f = 2$ phase of QCD with three flavors is $\Delta_0(N_f=3) = 47$ MeV. This large reduction comes from the factor $N_f^{-5/2}$ in the perturbative expression for Δ_0. This factor is a reflection of the N_f dependence of the screening mass. The color anti-symmetric and symmetric order parameters in the color-flavor locked phase are $\Delta_A = 38$ MeV and $\Delta_S = 1.3$ MeV. The condensation energy is $\epsilon = -32\,\mathrm{MeV/fm}^3$. The superfluid order parameter is given by $\phi_A = (144\,\mathrm{MeV})^3$. From this we find that the chiral condensate is $\langle\bar\psi\psi\rangle = -(35\,\mathrm{MeV})^3$, while the expectation value of the four fermion operator is significantly bigger $\langle(\bar\psi\psi)^2\rangle = (194\,\mathrm{MeV})^6$.

VI. CONCLUSIONS

In this work we studied the structure of QCD with an arbitrary number of flavors at high baryon density. We assumed that the dominant instability of the quark fermi liquid is

towards the formation of a pair condensate. In order to study the ground state, we introduced a renormalized mean field grand canonical potential which determines the condensation energy as a function of the color-flavor structure of the gap matrix. This potential is not derived from QCD, but we expect it to correctly represent the essential dynamics of the high density phase. From the analysis of the grand potential we find that for any number of flavors greater than two, chiral symmetry is broken while a vector-like flavor symmetry remains. The most interesting case is QCD with $N_f = 3$ flavors. In this case, the symmetry breaking pattern at high density matches the one at low density. The color-flavor locked state in $N_f = 3$ is also distinguished by the fact that it has the biggest condensation energy per flavor, and that it leaves the largest subset of the original flavor symmetry intact.

In the second part of this work we studied the $N_f = 3$ phase in more detail. We calculated the color symmetric and anti-symmetric order parameters in perturbative QCD. We showed that Δ_S is suppressed by one power of g in the weak coupling limit, and that the color-flavor locked state is the energetically favored state. We calculated $\langle\bar{\psi}\psi\rangle$ and $\langle(\bar{\psi}\psi)^2\rangle$. We explicitly showed that the chiral condensate is suppressed because of an approximate $Z_2 \times Z_2$ symmetry. In general, the expectation value of four-fermion operators is not suppressed, but all gauge invariant four-fermion chiral order parameters vanish in the mean field approximation. Non-vanishing gauge invariant chiral order parameters can be obtained by considering higher dimension operators, semi-classical (instanton) effects, or higher order perturbative corrections.

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VII. ERRATUM

There is a mistake in equs. (11) and (12) in which we describe the symmetry breaking pattern in the case $N_c = 3$, $N_f = 4$. The second set of $SU(2)$ generators given in equ. (12) does not generate a symmetry of the order parameter (and it does not commute with the

first set of generators). The correct generators are

$$\left(\epsilon^{abc}N_{bc}+\frac{1}{2}\eta^a_{\mu\nu}M^{\mu\nu}\right),\qquad\left(\frac{1}{2}\bar{\eta}^a_{\mu\nu}M^{\mu\nu}\right). \tag{51}$$

Note that the second $SU(2)$ is a pure flavor symmetry. As a consequence, the symmetry breaking pattern is

$$SU(4)_L\times SU(4)_R\times U(1)_V\to SU(2)_V\times SU(2)_V\times SU(2)_A \tag{52}$$

and the axial $SU(4)_A$ symmetry is not completely broken. I thank H. Murayama for pointing out this error.

There is a factor 2 mistake in equation (31) for the thermodynamic potential. To $o(g^2)$ the thermodynamic potential is given by

$$\begin{aligned}\Omega &= \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left\{-\mathrm{tr}\left[S(q)\Sigma(q)\right]+\mathrm{tr}\log\left[S_0^{-1}(q)S(q)\right]\right\} && (53)\\ &+\frac{1}{4}\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[S(q)\Gamma^a_\mu S(q+k)\Gamma^b_\nu\right]D^{ab}_{\mu\nu}(k), && (54)\end{aligned}$$

where Γ^a_μ is the quark gluon vertex function. The last term can be simplified using the gap equation. We find

$$\Omega=\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left\{-\frac{1}{2}\mathrm{tr}\left[S(q)\Sigma(q)\right]+\mathrm{tr}\log\left[S_0^{-1}(q)S(q)\right]\right\}. \tag{55}$$

Note that the first term differs from equ. (31) by a factor 1/2. As a consequence, equ. (32) should read

$$f(\Delta)=\frac{\mu^2}{\pi^2}\int dp_0\left\{-\frac{\Delta(p_0)^2}{2\sqrt{p_0^2+\Delta(p_0)^2}}+\sqrt{p_0^2+\Delta(p_0)^2}-p_0\right\}. \tag{56}$$

and equ. (33) is replaced by

$$f(\Delta)=\frac{\mu^2}{4\pi^2}\Delta_0^2, \tag{57}$$

which is identical to the result in BCS theory.

Finally, equ. (48) gives an estimate of the quark condensate in the CFL phase. The complete weak coupling result is

$$\langle\bar{\psi}\psi\rangle=-2\left(\frac{\mu^2}{2\pi^2}\right)12G\phi_A^2, \tag{58}$$

which was obtained in T. Schäfer, Phys. Rev. D65, 094033 (2002).

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N_c	N_f	N_{par}	gaps (deg)	Δ	$-\epsilon/(N_c N_f)$	N_{sym}	rank
3	2	3	Δ (4), 0 (2)	Δ_0	ϵ_0	6	2
3	3	9	Δ (8), 2Δ (1)	$0.80\Delta_0$	$1.27\epsilon_0$	8	2
3	4	18	Δ (8), 2Δ (4)	$0.63\Delta_0$	$1.21\epsilon_0$	6	2
3	5	30	Δ (5), 2Δ (7), 3Δ (3)	$0.43\Delta_0$	$1.18\epsilon_0$	3	1
3	6	45	Δ (16), 2Δ (2)	$0.80\Delta_0$	$1.27\epsilon_0$	9	3

TABLE I: Spectrum and symmetry properties of the s-wave superfluid state in QCD with $N_c = 3$ colors and N_f flavors. $N_{par} = N_c(N_c-1)N_f(N_f-1)/4$ is the number of totally anti-symmetric gap parameters. The fourth column gives the relative magnitude of the gaps in the fermion spectrum, together with their degeneracy. The next two columns give the magnitude of the gap and the condensation energy per species in units of the $N_f = 2$ values. These ratios are independent of the coupling in the weak-coupling limit. The last two columns show the dimension and the the rank of the residual symmetry group.